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Envelope solitons in inhomogeneous media

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Abstract. A search is made for a transformation that reduces the nonlinear Schrödinger equation, when an inhomogeneous medium is considered, to the usual one. It is found that such a transformation can be found for certain quadratic inhomogeneities.

1. Introduction

The study of the propagation of nonlinear waves in dispersive and inhomogeneous media is of practical interest in a wide range of plasma phenomena, as well as in other branches of physics.

When an almost monochromatic wave is considered in a strongly dispersive, weakly nonlinear, and nonuniform medium, a nonlinear Schrödinger equation arises (Chen and Liu 1978). If the density profile, in units of the unperturbed density n_0, is taken as $n(x)=1+s(x)$, where $s(x)$ is an arbitrary inhomogeneity, we get in dimensionless units

$$\mathrm{i}q_t+q_{xx}+(|q|^2-s(x))q=0. \tag{1.1}$$

If the medium were homogeneous, i.e. $s(x)=0$, then (1.1) would be the usual nonlinear Schrödinger equation (NLS), which has multisoliton solutions, and whose initial value problem may be solved through the inverse scattering method (Zakharov and Shabat 1972). Chen and Liu (1976, 1978) have found that if $s(x)$ is linear, the inverse scattering method is still applicable, and furthermore, that for $s(x)=L(t)+xM(t)$ in general, it is possible to find a transformation that reduces (1.1) to the usual NLS. The slope of the inhomogeneity accelerates the solitons as classical particles. This result may be used to study the behaviour of a single soliton in an arbitrary inhomogeneity, provided its width is much smaller than the scale of the inhomogeneity. This is clear, since under such adiabatic approximation, the soliton essentially feels a linear density profile at a given time t.

If $s(x)$ is complex, collisional damping is also taken into account by the imaginary part. The propagation of a single soliton in this case has been studied numerically for a linear inhomogeneity (Morales and Lee 1974), and analytically for a parabolic density profile (Gupta *et al* 1978). In the latter work it was found that, for a certain relation between the quadratic term and the damping coefficient, the equation can be solved exactly.

The purpose of this work is to study under what circumstances it is possible to find a transformation that reduces (1.1) to the usual NLS, given an arbitrary inhomogeneity

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$s(x)$. The result of Chen and Liu (1978) is obtained as a particular case. In § 2, a transformation is proposed that yields a set of partial differential equations. It is shown from these equations that the most general form of $s(x)$ that the transformation admits is a quadratic one. Some examples are shown. Section 3 is devoted to concluding remarks.

2. The transformation

We wish to find a transformation such that (1.1) may take the form

$$iq_t + q_{xx} + (|q|^2 - s(x))q = [\alpha(iu_\tau + u_{yy} + |u|^2 u) + F_1(\alpha, A)u_y + F_2(\alpha, A, s)u]e^A = 0, \tag{2.1}$$

where $\alpha = \alpha(y, \tau)$, $u = u(y, \tau)$ and $A = A(y, \tau)$, so that by imposing $F_1 = 0 = F_2$ we may arrive at the reduction we are looking for. With this in mind, we propose

$$q(x, t) = \alpha^{1/3}(y, \tau)u(y, \tau)\, e^{A(y,\tau)} \tag{2.2}$$

and

$$x = \beta(y, \tau), \qquad t = \gamma(y, \tau), \tag{2.3}$$

where α, β, γ and A are to be determined by $s(x)$. Since we ignore the forms of β and γ, let us take

$$\partial/\partial x = \alpha^{1/3}\, \partial/\partial y + \delta(\alpha)\partial/\partial\tau, \tag{2.4a}$$

$$\partial/\partial t = \varepsilon(\alpha)\, \partial/\partial y + \alpha^{2/3}\, \partial/\partial\tau. \tag{2.4b}$$

When this transformation is applied, (2.1) will include the undesired term $\delta^2\alpha^{1/3}u_{\tau\tau}\exp(A)$, which means we must take $\delta(\alpha) = 0$ from the very beginning. Thus, by inverting (2.4) we find that

$$\beta_y = \alpha^{-1/3}, \qquad \gamma_y = 0, \qquad \beta_\tau = -\alpha^{-1}\varepsilon(\alpha), \qquad \gamma_\tau = \alpha^{-2/3}. \tag{2.5}$$

Imposing $F_1 = 0$ and $F_2 = 0$, we get

$$\alpha_y - i\alpha^{4/3}\beta_\tau + 2\alpha A_y = 0, \tag{2.6}$$

and

$$(\alpha\, iA_\tau + A_{yy} - A_y^2) + \tfrac{1}{3}i\alpha_\tau + \tfrac{1}{3}\alpha_{yy} - (\tfrac{1}{9}\alpha^{-1}\alpha_y - \tfrac{1}{3}i\beta_\tau\alpha^{1/3})\alpha_y - \alpha^{1/3}s = 0, \tag{2.7}$$

respectively. These two equations must be solved together with (2.5). From the latter, we find that $\gamma_{y\tau} = \alpha^{-5/3}\alpha_y = 0$, so the only nontrivial case we are left with is $\alpha_y = 0$. Thus, (2.6) and (2.7) become

$$A_y = \tfrac{1}{2}i\alpha^{1/3}\beta_\tau, \tag{2.8}$$

$$A_\tau = iA_{yy} - iA_y^2 - \tfrac{1}{3}\alpha^{-1}\, d\alpha/d\tau - i\alpha^{-2/3}s. \tag{2.9}$$

On the other hand, β_y may be integrated,

$$\beta = \alpha^{-1/3}y + c(\tau) \tag{2.10}$$

where $c(\tau)$ is an arbitrary function of τ. Substituting (2.8) and (2.10) in (2.9), it is found that

$$(A_y)_\tau-(A_\tau)_y=\left[\frac{i}{9}\alpha^{-2}\left(\frac{d\alpha}{d\tau}\right)^2-\frac{i}{6}\alpha^{-1}\frac{d^2\alpha}{d\tau^2}\right]y$$

$$+\frac{i}{3}\alpha^{-2/3}\frac{d\alpha}{d\tau}\frac{dc}{d\tau}+\frac{i}{2}\alpha^{1/3}\frac{d^2c}{d\tau^2}+i\alpha^{-2/3}s_y=0. \qquad (2.11)$$

This means that $s_{yyy}=0$, so the most general form that s may have is the one of a quadratic polynomial,

$$s(x)=L(t)+M(t)x+N(t)x^2=L[\gamma(\tau)]+M[\gamma(\tau)]\beta(y,\tau)+N[\gamma(\tau)]\beta^2(y,\tau). \qquad (2.12)$$

This may be also seen directly from the fact that $\beta_{yy}=0$. Thus, (2.10) may be used to find s_y, and (2.11) yields the following set of ordinary differential equations in $\alpha(\tau)$ and $c(\tau)$:

$$d^2\alpha/d\tau^2-\tfrac{2}{3}\alpha^{-1}(d\alpha/d\tau)^2-12\alpha^{-1/3}N=0, \qquad (2.13)$$

$$\tfrac{1}{2}\alpha\, d^2c/d\tau^2+\tfrac{1}{3}(d\alpha/d\tau)dc/d\tau+\alpha^{-1/3}(M+2cN)=0. \qquad (2.14)$$

If $N=0$, then α may be taken to be 1. As a consequence, $t=\gamma=\tau$, and (2.14) yields

$$c(t)=-2\int_0^t\int_0^{t'}M(t'')\,dt''\,dt'. \qquad (2.15)$$

This is the result previously obtained by Chen and Liu (1978).

In order to solve (2.13) for the general case, we may take $Y=\alpha^{1/3}$. It transforms into

$$Y^3\,d^2Y/d\tau^2=4N, \qquad (2.16)$$

which may be solved by a new change of variable

$$u=d^2Y/d\tau^2, \qquad Y=\tfrac{1}{2}u\tau^2, \qquad (2.17)$$

where integration constants have been taken to be zero for simplicity. More general solutions may be obtained otherwise. In this case, if N is a constant, we get the following solutions for α, β and γ:

$$\alpha=(2N)^{3/4}\tau^{3/2}, \qquad (2.18)$$

$$\beta=(2N)^{-1/4}\tau^{-1/2}y+c(\tau), \qquad (2.19)$$

$$\gamma=(2N)^{-1/2}\ln(\tau). \qquad (2.20)$$

The form of $c(\tau)$ should be obtained by solving (2.14), which now has the form

$$d^2c/d\tau^2+\tau^{-1/2}\,dc/d\tau+2\tau^{-2}c=-\tau^{-2}M(\gamma)N^{-1}. \qquad (2.21)$$

If $M(\gamma)=0$, one may take the trivial solution $c=0$. The case $M(\gamma)\neq 0$ need not bother us, since it is always possible to reduce $s(x)$ to the form $s(x)=L(t)+Nx^2$ by choosing the origin appropriately. The form of A may be obtained integrating (2.8):

$$A=-\tfrac{1}{8}i\tau^{-1}y^2+T(\tau), \qquad (2.22)$$

where $T(\tau)$ is a function of τ which depends on $c(\tau)$. If it is taken to be zero, as in this case, $T(\tau)=0$.

Solving (2.16) for the more general case in which $N(\gamma)$ is not a constant may prove to be extremely difficult; however, some cases in which this is readily possible may be easily identified. For instance, if $N[\gamma(\tau)] = (k^{4/3}/2)\tau^{4n+6}$, where k is an arbitrary constant and n is an integer,

$$\alpha = k\tau^{3(n+2)}, \tag{2.23}$$

$$\beta = k^{-1/3}\tau^{-(n+2)} + c(\tau), \tag{2.24}$$

$$\gamma = [k^{-2/3}/(-2(n+2)+1)]\tau^{-2(n+2)+1}. \tag{2.25}$$

In this case again, if the origin is chosen in such a way that $M(\gamma) = 0$, we may take $c = 0$. From (2.8),

$$A = -\tfrac{1}{4}\mathrm{i}(n+2)^{-1}y^2. \tag{2.26}$$

It is also possible to find the forms of $M(t)$ and $N(t)$ that may allow the reduction (2.1), starting with a given $\alpha(\tau)$. The transformation β is found from (2.10), and γ from (2.5). Then $N(t)$ is obtained from (2.13) and $M(t)$ from (2.14), where we also need to choose $c(\tau)$.

3. Conclusion

It has been found that (1.1) can be reduced to the usual NLS equation using a transformation of the form (2.2) if the inhomogeneity $s(x)$ is quadratic. The reason why it is not possible to consider more general forms of $s(x)$ is that $\delta(\alpha) = 0$ in (2.4a), and this implies subsequently that $\gamma_y = 0$, then $\beta_{yy} = 0$ and finally $s_{yyy} = 0$.

The problem of finding the appropriate transformation is essentially reduced to the one of solving (2.13) and (2.14).

A different approach to the problem addressed in this work may be to follow the ideas of Balakrishnan (1982), where the application of the inverse scattering method is attempted, by extending the AKNS formalism (Ablowitz *et al* 1974). The main idea is to allow the eigenvalue ζ to depend upon x and t. It may be readily seen that if $\zeta = \zeta(t)$, it will be possible to deal with linear inhomogeneities. This is essentially the result of Chen and Liu (1976). If more general forms of $s(x)$ are considered, however, ζ must have a full dependence upon x and t.

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